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Technical Note

Modeling the effects of a magnetic field or rotation on flow in a porous medium: momentum equation and anisotropic permeability analogy

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Abstract

It is pointed out that, in recent publications, several authors have incorrectly omitted a porosity factor, from a term modeling the effect of a magnetic field or rotation, in the momentum equation modeling flow in a porous medium. The error is linked with the way in which the pressure is incorporated in the standard differential form of Darcy's law, and this is discussed in detail. A new form for this equation is proposed. Also, the analogy between: (i) Darcy flow in an isotropic porous medium with a magnetic field or rotation effect present; and (ii) flow in a medium with anisotropic permeability, is discussed.

1. Introduction

The paper by Yih [1] is remarkable because, despite the fact that it is on MHD (magnetohydrodynamic) mixed convection, all the results reported are for the case of zero Hartmann number, which is the case where the MHD effect is absent. However, a much more important feature of the paper is that the effect of the magnetic field has been modelled incorrectly. In common with many other authors who have written on the effects of a magnetic field or rotation on flow in a porous medium, the author of [1] has overlooked a subtlety which is pointed out and discussed in Section 2. The same paper [1] serves to illustrate an analogy between on the one hand flow in an isotropic porous medium subject to the effect of a magnetic field, or the effect of rotation, and on the other hand flow in a porous medium with anisotropic permeability, for the case of Darcy flow. This analogy is discussed in Section 3.

2. The momentum equation

Eq. (2) of [1] reads

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial P}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}-\frac{\nu}{K}u-\frac{\sigma B_0^2}{\rho}u\pm g\beta(T-T_\infty) \qquad (1)$$

where u and v are the components of the velocity in the x and y directions respectively, P is the pressure, ρ is the density, ν is the kinematic viscosity, K is the permeability of the porous medium, σ is the electrical conductivity, B_0 is the externally imposed magnetic field in the y-direction, g is the gravitational acceleration, T is the temperature and T_∞ is the free stream temperature, in this boundary layer approximation. The author of [1] does not distinguish between the Darcy (seepage)

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velocity (average over a representative elementary volume (REV) of the porous medium) and the intrinsic velocity of the fluid (average over the fluid portion of the REV). The Darcy velocity equals the intrinsic velocity multiplied by ϕ, the porosity of the porous medium. This is the Dupuit–Forchheimer relationship. It is important to note that in the standard formulation of Darcy's law (whereby the permeability K is defined) it is the fluid pressure (an intrinsic quantity) and the Darcy velocity which appear. Consequently, the velocity u which appears on the right-hand side of Eq. (1) has to be the Darcy velocity and ν represents an effective kinematic viscosity in the second term on the right hand side. On the other hand, the magnetic drag term (the second to last term) involves the intrinsic velocity (since that drag is a body force on the moving fluid, and not on the stationary solid matrix of the porous medium). If u is defined to be the Darcy velocity, then the magnetic drag term requires a factor ϕ^{-1}. Likewise, it is the intrinsic velocity which is involved in the inertia terms, and so both terms on the left-hand side must be multiplied by a factor ϕ^{-2}. (Alternatively, if u is defined as the intrinsic velocity, K has to be replaced by K/ϕ.) The use of the wrong equation is of long standing. It appears, for example, in the review by Raptis and Perdikis [2].

In the case of the effect of rotation, expressed as a Coriolis force, a similar argument applies. It is the inertial force on the fluid, and not the fluid–solid composite, which is involved, and hence it is the intrinsic velocity which appears, multiplied by the angular velocity, in the Coriolis term which is added to the momentum equation. Many authors have overlooked the fact that a factor ϕ^2 should appear in the denominator of the Taylor number (or, equivalently, a factor ϕ in the numerator of the Ekman number) which they use. For example, the factor is missing in [3], but Vadasz in his review [4] and other recent papers has the correct factor.

The interpretation of the pressure in the Darcy equation as an intrinsic quantity is consistent with the way in which a buoyancy term is added in combination with the pressure gradient term; the porosity is not involved. Here Eq. (1) serves as an example. The published literature is in consistent agreement on this point. If the pressure were an REV average, then a factor ϕ would be involved in the buoyancy term (see below).

It is worthwhile considering in detail why the pressure in the traditional Darcy differential equation is necessarily an intrinsic quantity rather than a seepage quantity. The reader should note that the original Darcy equation (relating to Darcy's experiments), in the form 'pressure drop divided by length of column equals constant times seepage velocity', leads directly to the modern differential equation for sufficiently slow flow. For the purpose of deriving this differential equation, the REV is properly regarded as a miniature column to which Darcy's result can be applied. As well as the assumptions of steady flow and incompressible fluid, three other important assumptions are made.

1. The porous medium is assumed to be homogenous.
2. The macroscopic (REV scale) flow is assumed to be unidirectional.
3. A continuum assumption is made.

A consequence of the first assumption is that no distinction need be made between surface porosity and volume porosity. A consequence of the second assumption is that there is no flow out of the sides of the miniature column, so the situation in that column is the same as in the large column. A consequence of the third assumption is that it is permissible to consider the mathematical limit as the length of the miniature column tends to zero. In Darcy's experiments the pressure drop was a *fluid* pressure drop measured in the usual way by a pair of manometers placed outside the porous medium. Each manometer measured the fluid pressure at the point at which it was placed and no cross-sectional average was involved. Rather, because of assumptions 1 and 2 above, the pressure could be regarded as uniform, and so the pressure measured at one point was representative of the whole cross-section occupied by fluid, so the pressure measured by Darcy is an intrinsic pressure. It follows that, after the appropriate limit of length of the miniature column tending to zero is taken, the pressure in the modern differential equation is also an intrinsic quantity. This means that the pressure gradient at the REV level is effectively the average of the microscopic (pore scale) fluid pressure gradient averaged over just the fluid portion of the REV.

(Professor J. L. Lage has pointed out to the author in a personal communication that the fact that Darcy measured pressure just outside the porous medium rather than just inside it means that an entrance and exit effect is involved, and this affects the permeability value measured in his experiments. In the argument presented here it is assumed that this effect is negligible.)

Conversely, one can start with the differential equation and deduce an expression for the Darcy pressure drop. Because of assumptions 1 and 2, macroscopic transverse pressure gradients are zero. Further, by mass conservation, the seepage velocity is independent of the longitudinal coordinate, and so the same must be true of the longitudinal pressure gradient. For an integration over a volume, one can treat the medium as a continuum, in which no distinction need be made between the fluid and solid phases. That means that mathematically one can average over the entire cross-section of the Darcy column, and integrate between the ends of the column, and thereby recover the original Darcy expression for the pressure drop. In fact, since the pressure gradient is uniform over the whole column, the mathematics involved is very simple. In this process one starts with a fluid pressure gradient and ends up with a fluid pressure drop. Thus the argument is consistent.

There is overwhelming experimental evidence that the conventional differential equation, with P denoting an intrinsic quantity and K the standard permeability, is correct. If P were an REV averaged quantity but with K unchanged, then a factor ϕ would have to appear in the buoyancy term in the momentum equation, and consequently in the definition of Rayleigh number (for example, in the last term of Eq. (6.4) and in (6.19) of the book by Nield and Bejan [5]), and then there would be a discrepancy with the well established experimental value for the critical Rayleigh number for the Horton–Rogers–Lapwood problem (see Section 6.9.1 of [5]), to give just one example. Elder [6] obtained the experimental value 40 with an estimated experimental error of 10%, in comparison with the theoretical value of 39.48. In his experiment he used a bed of packed spheres, and so the porosity was about 0.4. A critical Rayleigh number of 40/0.4 is incompatible with the experimental results.

When the conventional definition of permeability was introduced about 1920 (for references, see [7]), a definition which was popularized by the writings of Muskat [8], workers had the option of invoking the Dupuit–Forchheimer relationship and expressing Darcy's law in terms of intrinsic velocity, effectively defining a different permeability which incorporates the factor ϕ, but they did not do so. The mixture of intrinsic pressure and seepage velocity in the conventional equation is unfortunate. Lage [7] has recognized this, and has written an equation in terms of a seepage pressure and seepage velocity. The author now proposes what he believes is a better alternative, namely to write the entire equation in terms of intrinsic quantities. In order to avoid possible confusion,we will introduce a new quantity with a different name, symbol and dimensions from the permeability K. The Darcy differential equation is written in the form

$$\nabla P + \mu R\mathbf{V} = 0. \qquad (2)$$

where R denotes the 'retardability', which has dimensions (length)$^{-2}$ and is thus measured in terms of the unit m^{-2}, and is defined in terms of the standard permeability K and porosity ϕ by $R = \phi/K$, while P is the intrinsic pressure and $\mathbf{V}$ is the intrinsic velocity. The major advantage of the new form is that it generalizes in a natural way to the Brinkman equation,

$$\nabla P + \mu R\mathbf{V} - \mu_\epsilon \nabla^2 \mathbf{V} = 0. \qquad (3)$$

where μ_ϵ is an effective viscosity. Volume averaging over an REV gives the estimate $\mu_\epsilon = \mu$ (rather than μ/ϕ). Clearly, the porosity ϕ does not appear explicitly in the equation, but is incorporated into the geometrical factor R. Other intrinsic terms, like the Coriolis term can be added and expressed in terms of $\mathbf{V}$, and again the porosity does not appear in these terms. For the case of buoyancy, the term to be added to the left hand side of Eq. (3) is $-\rho\boldsymbol{g}$, where $\boldsymbol{g}$ is the gravitational acceleration. A minor bonus is that the division solidus does not appear in the equation. The new form of the equation should be especially convenient when modeling hyperporous media [9], those for which the Darcy number is not small compared with unity, and so the Darcy and Brinkman resistance terms are of the same order of magnitude throughout the porous medium. As Nield and Lage [9] pointed out, in such media the permeability cannot be determined in a simple way from a Darcy type experiment.

For the Brinkman–Forchheimer equation the following form is proposed:

$$\nabla P + \mu R\mathbf{V} - \mu_\epsilon \nabla^2 \mathbf{V} + C_N \rho R^{1/2} V\mathbf{V} = 0. \qquad (4)$$

The new nondimensional Forchheimer coefficient C_N is related to the old Forchheimer coefficient C_F used in [5] by $C_N = \phi^{3/2} C_F$. There is evidence which suggests that C_N may be closer to being a universal constant than is C_F. For example, Beavers and Sparrow [10] noted that for their fibrous foam metal materials C_F was about 0.1, compared with the value 0.55 obtained with beds of spheres [11]. Unfortunately the porosity values are not reported in [10], but plausible ballpark values are 0.9 for the fibrous materials and 0.4 for the beds of spheres, and these values yield the C_N values 0.085 for the fibrous material and 0.14 for the beds of spheres.

One should note that there are difficulties in deriving the Darcy equation by considering a force balance over an REV considered as a control volume, because not all of the material in the control volume is free to move, and an indeterminate solid–solid contact force is involved at a solid portion of the boundary of an REV. Rather, the Darcy equation is best regarded as a statement about the flow of fluid relative to the solid matrix, which provides a resistance to the flow. The paper by Fulks et al. [12] presents a correct and relatively simple derivation of a Darcy momentum equation. This is based on considerations of force balance applied to the region occupied by fluid.

3. An analogy

There is a further noteworthy feature of Eq. (1). The Darcy drag term and the magnetic drag term are both linear in the velocity, and so can be combined into a single term. With the correct porosity factor included, one has a total drag

$$\left(\frac{\nu}{K}+\frac{\sigma B_0^2}{\phi\rho}\right)u=\frac{\nu}{K^*}u, \tag{5}$$

where

$$K^*=\frac{K}{1+(\sigma B_0^2K/\phi\mu)}=\frac{K}{1+N}, \tag{6}$$

where $\mu=\rho\nu$ is the dynamic viscosity. The new dimensionless parameter

$$N=\sigma B_0^2K/\phi\mu=\sigma B_0^2/\mu R \tag{7}$$

which appears here may be interpreted as the ratio of rate of magnetic dissipation (of mechanical energy, per unit volume) to rate of viscous dissipation.

Thus the effect of the uniform applied magnetic field, directed perpendicular to the boundary, is to reduce the effective permeability in this case of boundary layer flow. In the more general situation, the main hydrodynamic effect of a uniform applied magnetic field is to inhibit flow transverse to the magnetic field lines but not flow longitudinal to the magnetic field lines.Consequently, there is an analogy between the MHD flow in an isotropic medium and hydrodynamic flow in a medium with anisotropic permeability, the principal axes of anisotropy being aligned parallel and perpendicular to the uniform applied magnetic field. The longitudinal permeability is not affected by the magnetic field, but the transverse permeability is decreased by a factor $(1+N)^{-1}$.

In the case of flow in a rotating porous medium the situation is similar. Flow transverse to the rotation vector is inhibited (the Taylor–Proudman effect) while flow longitudinal to the rotation vector is not inhibited. As a result, Darcy flow in a rotating isotropic porous medium is analogous to flow in a medium with anisotropic permeability. This analogy was demonstrated by Palm and Tyvand [3], in the context of the onset of natural convection induced in a horizontal layer uniformly heated from below. The author is not aware of any previously published discussion of the corresponding MHD analogy.

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